Math 245C Lecture 21 Notes

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1 Isomorphism, Unitary Property of the Fourier Transform, and Periodic Functions

1.1 The Fourier transform on the Schwarz space

If $f, \widehat{f} \in L^1$, then $f \overset{\text{a.e.}}{=} (f^\vee)^\wedge$, where, $f^\vee = \widehat{f} \circ O$, and $O(x) = -x$.

Corollary 1.1. *If $f \in L^1$ and $\widehat{f} = 0$, then $f \equiv 0$ a.e.*

Proof. We have $f, \widehat{f} \in L^1$, and so

$$f \equiv (f^\vee)^\wedge = (\widehat{f} \circ 0)^\wedge = 0^\wedge = 0. \qquad \square$$

Corollary 1.2. *$\mathcal{F} : \mathcal{S} \to \mathcal{S}$ is an isomorphism.*

Proof. By the previous corollary, the kernel of $\mathcal{F}|_{\mathcal{S}}$ is $\{0\}$. Since $\mathcal{F}$ is linear, we conclude that $\mathcal{F}|_{\mathcal{S}}$ is one-to-one. We want to show that $\mathcal{F}|_{\mathcal{S}}$ is onto. Let $g \in \mathcal{S}$. Since $\widehat{g} \in \mathscr{S}$, we have $\widehat{g}, g \in \mathcal{S}$, and so $g = \widehat{\widehat{g} \circ O} = \mathcal{F}(\widehat{g} \circ O)$. Since $\widehat{g} \cong O \in \mathcal{S}$, we have proven that $\mathcal{F}^{-1}(g) = \widehat{g} \circ O$. That is, $\mathcal{F}^{-1} = \mathcal{F} \circ O$. Since $\mathcal{F}$ maps $\mathcal{S}$ continuous to $\mathcal{S}$, so does $\mathcal{F} \circ O = \mathcal{F}^{-1}$. $\square$

1.2 Unitary property of the Fourier transform

Theorem 1.1. *The Fourier transform has the following properties:*

1. *$\mathcal{F}$ maps $L^1 \cap L^2$ into L^2.*
2. *$\mathcal{F}$ extends to a unitary transformation $\tilde{\mathcal{F}} : L^2 \to L^2$.*

Proof. Set $A = \{f \in L^1 : \widehat{f} \in L^1\}$. We claim that $A \subseteq L^2$. Let $f \in A$. Then $f = (f^\vee)^\wedge$ a.e. This is in L^∞, as $\widehat{f} \in L^1$. Since $\frac{1}{2} = \frac{1/2}{1} + \frac{1/2}{\infty}$, we conclude that

$$\|f\|_2 \leq \|f\|_\infty^{1/2} \|f\|_1^{1/2}.$$

Observe that $L^2 = \overline{\mathcal{S}}^{L^2} \subseteq \overline{A}^{L^2} \subseteq L^2$. So A is dense in L^2.

Isometry: Let $f, g \in A$. We have

$$\int_{\mathbb{R}^n} f\overline{g} = \int_{\mathbb{R}^n} f(\overline{g}^\vee)^\wedge = \int_{\mathbb{R}^d} \widehat{f}\overline{g}^\vee = \int \widehat{f}\overline{\widehat{g}}.$$

In particular,

$$\int_{\mathbb{R}^n} |f|^2\,dx = \int_{\mathbb{R}^n} |\widehat{f}|^2\,d\xi.$$

Extension: Since A is dense in L^2, this gives us that f extends to a linear operator $\tilde{\mathcal{F}} : L^2 \to L^2$ such that $\|\tilde{\mathcal{F}}\|_2 = \|f\|_2$.

It remains to check that $\tilde{\mathcal{F}} = \mathcal{F}(f)$ for $f \in L^1 \cap L^2$. Set

$$\rho(x) = e^{-\pi|x|^2}, \qquad \rho_r(x) = \frac{1}{t^n}\rho(x/t).$$

Let $f \in L^1 \cap L^2$. We have $\rho_t * f \in L^1 \cap L^2$, and

$$\widehat{\rho_t * f} = \widehat{\rho}_t\widehat{f} = \underbrace{e^{2\pi i|\xi|^2}}_{\in L^1}\underbrace{f(\xi)}_{\in L^\infty}.$$

So $\widehat{\rho}_t * f \in L^1$. This means that $\rho_t * f \in A$. We have that

$$\|\mathcal{F}(\rho_t * f) - \mathcal{F}(f)\|_2 = \|\tilde{\mathcal{F}}(\rho_t * f) - \tilde{\mathcal{F}}(f)\|_2 = \|\rho_t * f - f\|_2,$$

$$\|\mathcal{F}(\rho_t * f) - \tilde{\mathcal{F}}(f)\|_\infty \le \|\rho_r * f - f\|_1.$$

Let $B \subseteq \mathbb{R}^n$ be a bounded ball. We have

$$\|\tilde{\mathcal{F}}(f) - \mathcal{F}(f)\|_2 \le \|\tilde{\mathcal{F}}(f) - \mathcal{F}(\rho_t * \mathcal{F})\|_2 + \|\mathcal{F}(\rho_t * f) - \mathcal{F}(f)\|_{L^2(B)} \le \|\tilde{\mathcal{F}}(f) - \mathcal{F}(\rho_t * \mathcal{F})\|_2 + \|\mathcal{F}(\rho_t * f) - \mathcal{F}(f)\|_\infty |B|$$

So we conclude that $\tilde{\mathcal{F}}(f) = \mathcal{F}(f)$ a.e. on B. □

Corollary 1.3. *For $1 \le p \le 2$ and $q = p/(p-1)$, we obtain an extension to $\mathcal{F} : L^p \to L^q$ such that $\|\mathcal{F}(f)\|_q \le \|f\|_p$.*

1.3 Producing periodic functions from L^1 functions

Theorem 1.2. *Let $f \in L^1$.*

1. *There exists a periodic $Pf : \mathbb{R}^n \to \mathbb{R}$ such that $\|Pf\|_1 \le \|f\|_1$.*
2. $\widehat{Pf}^{\mathbb{T}^n}(\ell) = \widehat{f}^{\mathbb{R}^n}(\ell)$.
3. $Pf(x) = \sum_{k \in \mathbb{Z}^n} \tau_k f(x)$.

Proof. Let $Q = [-1/2, 1/2)^n$. Set $F_m(x) = \sum_{|k|\le m, k\in\mathbb{Z}^n} f(x-k)$. By the monotone convergence theorem,

$$\int_Q \sum_{k\in\mathbb{Z}^n} |f(x-k)|\,dx = \sum_{k\in\mathbb{Z}^n} \int_Q |f(x-k)|\,dx = \sum_{k\in\mathbb{Z}^n} \int_{Q+k} |f(x)|\,dz = \int_{\mathbb{R}^n} |f(z)|\,dz.$$

This proves that the series $(F_m(x))_m$ converges absolutely for a.e. $x \in Q$. So $(F_m(x))_m$ converges for a.e. $x \in Q$ to a value $Pf(x)$. We have that Pf is periodic. We also get that

$$\|Pf\|_{L^1(Q)} \le \|f\|_1.$$

This completes the proofs of the first and third statements.

If $\ell \in \mathbb{Z}^n$, then

$$\begin{aligned}\widehat{Pf}^{\mathbb{T}^n}(\ell) &= \int_Q Pf(x) e^{-2\pi i\ell\cdot x}\,dx \\ &= \int_Q \sum_Q \sum_{k\in\mathbb{Z}^n} f(x-k) e^{-2\pi i\ell\cdot x}\,dx\end{aligned}$$

Let $z = x - k$.

$$\begin{aligned} &= \sum_{k\in\mathbb{Z}^n} \int_{Q+k} f(z) e^{-2\pi i\ell\cdot z} e^{-2\pi i k\cdot\ell}\,dz \\ &= \int_{\mathbb{R}^n} f(z) e^{2\pi i\ell\cdot z}\,dz = \widehat{f}(\ell). \end{aligned}$$

□